Cohomological Characterization of Vector Bundles on Grassmannians of Lines

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February 17, 2009

Abstract

We introduce a notion of regularity for coherent sheaves on Grassmannians of lines. We use this notion to prove some extension of Evans-Griffith criterion to characterize direct sums of line bundles. We also give a cohomological characterization of exterior and symmetric powers of the universal bundles of the Grassmannian.

Introduction

The notion of Mumford-Castelnuovo regularity of sheaves on the projective space, introduced in [11], has shown a very powerful tool, especially to study vector bundles. This theory allows to prove easily Horrocks criterion to characterize direct sums of line bundles as those bundles without intermediate cohomology, and its improvement by Evans-Griffith depending on the rank of the vector bundle. There have been several generalizations of this notion of regularity to other ambient spaces such as Grassmannians ([3]), products of projective spaces ([8], [5]) or quadrics ([2]). In most of the cases, the starting point is some variant of the Beilinson spectral sequence, so that the notion of regularity consists of a finite number of cohomological vanishings. Such a notion has a nice behaviour, in particular it can be proved that, if a coherent sheaf $\mathcal{F}$ is regular, so is any positive twist of it.

An easy approach to the Mumford-Castelnuovo regularity on the projective space is through the Koszul exact sequence, obtained from the Euler exact sequence. In fact, the definition of regularity of a sheaf can be done by imposing the vanishing of some cohomology of the terms appearing in the Koszul exact sequence twisted by the sheaf. In this paper, we explore this approach for Grassmannians of lines (the right generalization of the Koszul exact

*Supported in part by the Spanish Ministry of Education through the research project MTM2006-04785
Mathematics Subject Classification 2000: 14F05, 14J60.
Keywords: Universal Bundles on Grassmannians, Castelnuovo-Mumford regularity.

sequence becomes too complicated; this is why we concentrate in these particular Grassmannians). In order to make the theory to work well, we will need to impose in the definition the property that any positive twist of a regular sheaf is still regular. This means that our notion, consists of infinitely many cohomological vanishings. This will not be, however, a problem for the applications we have in mind (and in fact our definition include some vector bundles which are not regular in the sense of [3]).

We dedicate a first section to recall all the preliminaries we will need for the Grassmannians of lines, with special attention to the universal bundles and their cohomological properties. We will also determine the right generalization to these Grassmannians of the Koszul exact sequence.

In the second section we introduce our notion of regularity, which we will call G-regularity. We show that the natural candidates coming from the universal bundles are G-regular (Example 2.2) and we prove that this notion satisfies analogue properties to the Mumford-Castelnuovo regularity (Proposition 2.3). We also remark that, when $n = 2$, i.e. when the Grassmannian is a projective plane, G-regularity coincides with Mumford-Castelnuovo regularity (Example 2.4) and that, when $n = 3$, i.e. the Grassmannian is a quadric, the notion of G-regularity containd the stronger notion of regularity given in [2]. We finish this section with the first strong application of our theory: a generalization of Evans-Griffith criterion to characterize direct sums of line bundles (Theorem 2.6).

In the last section we prove our main results. We first give two criteria that, with a finite number of cohomological vanishings, imply that a vector bundle contains as a direct summand an exterior power of one universal bundle (Theorem 3.1) or a symmetric power of the other universal bundle (Theorem 3.2). With the same techniques of those results, we also give a cohomological characterization of those vector bundles that are direct sums of twists of the above exterior and symmetric powers (Theorem 3.3). In particular, for $n = 3$ we reobtain for the four-dimensional quadric the characterizion of the vector bundles without intermediate cohomology of [9], while for $n = 4$ we reobtain the characterization of direct sums of line bundles and twists of the universal bundles or their duals given in [1].

1 Preliminaries

Throughout the paper $\mathbf{P}^n$ will denote the projective space consisting of the one-dimensional quotients of the $(n+1)$-dimensional vector space V, while $G(1,n)$ (frequently denoted just by G) will be the Grassmann variety of lines in $\mathbf{P}^n$. We recall the universal exact sequence on $G = G(1,n)$:

$$0 \to S^\vee \xrightarrow{\varphi} V \otimes \mathcal{O}_G \xrightarrow{\psi} Q \to 0 \tag{1}$$

defining the universal bundles S and Q over G, of respective ranks $n-1$ and 2. We will also write $\mathcal{O}_G(1) = \bigwedge^2 Q \cong \bigwedge^{n-1} S$. In particular, we have natural isomorphisms

$$S^j Q^\vee \cong (S^j Q)(-j) \tag{2}$$

(where S^j denotes the j-th symmetric power) and

$$\bigwedge^{j} S^\vee \cong \bigwedge^{n-1-j} S(-1) \tag{3}$$

Recall that the Plücker embedding of G is defined by the quotient $\bigwedge^2 V \otimes \mathcal{O}_G \xrightarrow{\wedge^2\psi} \mathcal{O}_G(1)$, or equivalently by the quotient $\bigwedge^{n-1} V^* \otimes \mathcal{O}_G \xrightarrow{\wedge^{n-1}\varphi^t} \mathcal{O}_G(1)$.

The universal sequence (1) is the analogue in G of the Euler sequence in the projective space. The long Koszul exact sequence in the projective space comes by taking the top exterior product in the left map of the Euler sequence, while the taking smaller exterior products produces the Koszul exact sequence truncated at the left. In the case of Grassmannians of lines, for any $j \leq n-1$, taking the j-th exterior powers of φ in (1) produces a long exact sequence

$$0 \to \bigwedge^{j} S^\vee \to \bigwedge^{j} V \otimes \mathcal{O}_G \to \bigwedge^{j-1} V \otimes Q \to \cdots \to \bigwedge^{2} V \otimes S^{j-2}Q \to V \otimes S^{j-1}Q \to S^jQ \to 0. \tag{R_j}$$

Dualizing (R_j) and using the canonical isomorphisms (2) we get another exact sequence

$$0 \to S^jQ(-j) \to V^* \otimes S^{j-1}Q(-j+1) \to \cdots \to \bigwedge^{j-1} V^* \otimes Q(-1) \to \bigwedge^{j} V^* \otimes \mathcal{O}_G \to \bigwedge^{j} S \to 0 \tag{$R_j^\vee$}$$

Observe now that we can glue $(R^\vee_{n-1-j})$ twisted by $\mathcal{O}_G(-1)$ with (R_j) and, when $j = n-1$, we get the analogue of the Koszul exact sequence:

$$\begin{aligned} &0 \to \mathcal{O}_G(-n) \to \bigwedge^{n-1} V \otimes \mathcal{O}_G(-n+1) \to \bigwedge^{n-2} V \otimes Q(-n+1) \to \dots \\ \cdots \to \bigwedge^{2} V \otimes S^{n-3}Q(-n+1) &\to V \otimes S^{n-2}Q(-n+1) \to V^* \otimes S^{n-2}Q(-n+2) \to \dots \\ &\cdots \to \bigwedge^{n-2} V^* \otimes Q(-1) \to \bigwedge^{n-1} V^* \otimes \mathcal{O}_G \to \mathcal{O}_G(1) \to 0. \end{aligned} \tag{4}$$

As we will see, the relevant part of (4) is that the last morphism is the evaluation morphism for $\mathcal{O}_G(1)$, and that (4) defines an element in $\mathrm{Ext}^{2n-2}(\mathcal{O}_G(1), \mathcal{O}_G(-n)) = H^{2n-2}(\mathcal{O}_G(-n-1)) = H^{2n-2}(\omega_G)$, which is the Serre dual of the unit in $H^0(\mathcal{O}_G)$.

Remark 1.1. We recall that $\bigwedge^j S$ and S^jQ with $0 \leq j \leq n-2$ have no intermediate cohomology (we say that E on G has no intermediate cohomology if, for $i = 1, 2, \dots, 2n-3$ we have the vanishing $H^i_*(E) = 0$, i.e. $H^i(E(k)) = 0$ for each integer k). This is not the case for S^jQ with $j \geq n-1$. For example, the exact sequence $(R^\vee_{n-1})$ produces a nonzero element in $\mathrm{Ext}^{n-1}(\mathcal{O}_G(1), S^{n-1}Q(-n+1)) = H^{n-1}(S^{n-1}Q(-n))$. In fact this is the only nonzero intermediate cohomology of $S^{n-1}Q$, while (R_j) shows that the only nonzero intermediate cohomology of S^jQ with $j \geq n-1$ is $H^{n-1}(S^jQ(-n-k))$, with $k = 0, 1, \dots, j-n+1$ (observe that, by Serre duality and (2), it is enough to check the cohomology up to order $n-1$). We recall that, if $i \leq j$, there is a decomposition

$$S^iQ \otimes S^jQ = S^{i+j}Q \oplus (S^{i+j-2}Q)(1) \oplus (S^{i+j-4}Q)(2) \oplus \cdots \oplus (S^{j-i}Q)(i) \tag{5}$$

so that, again, the only nonzero intermediate cohomology of any $S^iQ \otimes S^jQ$ is $H^{n-1}(S^iQ \otimes S^jQ(-n-k))$ for some $k \geq 0$. Similarly, using this we deduce that, for any $i \leq n-2$, $\bigwedge^j S \otimes S^iQ$ has no intermediate cohomology except for

$$H^{n-1-j}(\bigwedge^{j} S \otimes S^{n-j-1}Q(-n+j)) = \mathrm{Ext}^{n-1-j}(S^{n-j-1}Q, \bigwedge^{j} S(-1))$$

(which is generated by the exact sequence (R_{n-1-j})) and

$$H^{2n-2-j}(\bigwedge^j S \otimes S^j Q(-n-1)) = \operatorname{Ext}^{2n-2-j}(S^j Q(n-j), \bigwedge^j S(-1))$$

(which is generated by the exact sequence obtained by glueing (R_{n-1-j}), $(R^\vee_{n-1-j})$ twisted by $\mathcal{O}_G(n-1-j)$ and (R_j) twisted by $\mathcal{O}_G(n-j)$.

In general, we will call unit element to the extension generating one of the above cohomological groups.

2 G-regularity and Evans-Griffith criterion on $G(1,n)$

Inspired by (4), we give the following definition:

Definition 2.1. We say that a vector bundle E on G is *G-regular* if, for any $k \geq 0$, the following conditions hold:

(i) $H^1(\mathcal{F} \otimes Q(k-1)) = H^2(\mathcal{F} \otimes S^2 Q(k-2)) = \cdots = H^{n-2}(\mathcal{F} \otimes S^{n-2} Q(k-n+2)) = 0$;

(ii) $H^{n-1}(\mathcal{F} \otimes S^{n-2} Q(k-n+1)) = H^n(\mathcal{F} \otimes S^{n-3} Q(k-n+1)) = \ldots$

$$\cdots = H^{2n-4}(\mathcal{F} \otimes Q(k-n+1)) = H^{2n-3}(\mathcal{F}(k-n+1)) = 0;$$

(iii) $H^{2n-2}(\mathcal{F}(k-n)) = 0$.

We will say that $\mathcal{F}$ is *m-G-regular* if $\mathcal{F}(m)$ is G-regular. We define the *G-regularity* of $\mathcal{F}$, *G-reg*$(\mathcal{F})$, as the least integer m such that $F(m)$ is G-regular. We set *G-reg*$(\mathcal{F}) = -\infty$ if there is no such an integer.

Example 2.2. We get from Remark 1.1 that the trivial bundle $\mathcal{O}_G$, any $\bigwedge^j S$ with $j \in \{1, \ldots, n-2\}$ or any $S^j Q$ are G-regular, and in fact their G-regularity is zero. This shows that the definition of Chipalkatti in [3] is much more restrictive than ours, since S is not regular with his definition.

On the other hand, if T is the tangent bundle of the Plücker ambient space of G, it follows from the restriction of the Euler exact sequence that $T_{|G}(-1)$ is G-regular, while $T_{|G}(-2)$ is not (because $H^{2n-3}(T_{|G}(-n-1)) \neq 0$), hence *$G$-reg*$(T_{|G}) = -1$.

We can now prove that our definition of regularity has the right properties one should expect:

Proposition 2.3. *If $\mathcal{F}$ is a G-regular coherent sheaf on $G = G(1,n)$ then, for any $k \geq 0$:*

(i) $\mathcal{F}(k)$ is G-regular.

(ii) $H^{2n-3}(\mathcal{F} \otimes \bigwedge^j S^\vee(k-n)) = 0$ for $j = 1, \ldots, n-2$, and $H^{n-2}(\mathcal{F} \otimes S^{n-1} Q(k-n+1)) = 0$.

(iii) For $j = 1, \ldots, n-1$, the multiplication map $H^0(\mathcal{F}(k)) \otimes H^0(\bigwedge^j S) \to H^0(\mathcal{F} \otimes \bigwedge^j S(k))$ is surjective.

(iv) The multiplication map $H^0(\mathcal{F}(k)) \otimes H^0(\mathcal{O}_G(l)) \to H^0(\mathcal{F}(k+l))$ is surjective for any $l \geq 1$.

(v) $\mathcal{F}(k)$ is generated by its global sections.

Proof. Part (i) comes from the definition of regularity. Part (ii) comes by taking cohomology in (R_j) tensored with $\mathcal{F}(n-k)$ for, respectively, $j = 1, \dots, n-1$. Part (iii) follows by taking cohomology in $(R_j^\vee)$ tensored with $\mathcal{F}$ and having in mind the identification $H^0(\bigwedge^j S) = \bigwedge^j V^*$.

We will prove (iv) by induction on l, the case $l = 1$ being (iii) for $j = n-1$. The statement for a general l comes from the commutative diagram

$$\begin{array}{ccc} H^0(\mathcal{F}(k)) \otimes H^0(\mathcal{O}_G(l-1) \otimes H^0(\mathcal{O}_G(1)) & \rightarrow & H^0(\mathcal{F}(k+l-1) \otimes H^0(\mathcal{O}_G(1)) \\ \downarrow & & \downarrow \\ H^0(\mathcal{F}(k)) \otimes H^0(\mathcal{O}_G(l)) & \rightarrow & H^0(\mathcal{F}(k+l)) \end{array}$$

using that the top map is surjective by induction hypothesis and the right map is surjective by applying again (iii) for $j = n-1$.

To prove (v), we consider a sufficiently large twist such that $\mathcal{F}(k+l)$ is generated by its global section. Consider the commutative diagram

$$\begin{array}{ccc} H^0(\mathcal{F}(k)) \otimes H^0(\mathcal{O}_G(l)) \otimes \mathcal{O}_G & \rightarrow & H^0(\mathcal{F}(k+l)) \otimes \mathcal{O}_G \\ \downarrow & & \downarrow \\ H^0(\mathcal{F}(k)) \otimes \mathcal{O}_G(l) & \rightarrow & \mathcal{F}(k+l) \end{array}$$

in which the top map is surjective by (iv) and the right map is surjective because $\mathcal{F}(k+l)$ is globally generated. This yields the surjectivity of $H^0(\mathcal{F}(k)) \otimes \mathcal{O}_G(l) \to \mathcal{F}(k+l)$, which implies that $\mathcal{F}(k)$ is generated by its global sections. $\square$

Example 2.4. If $n = 2$, then $G = G(1, 2)$ is a projective plane, and $\mathcal{F}$ is G-regular when, for any $k \geq 0$, $H^1(F(k-1)) = H^2(F(k-2)) = 0$, which coincides with the Castelnuovo-Mumford regularity on $\mathbf{P}^2$.

Example 2.5. If $n = 3$ then $G = G(1, 3)$ is a quadric hypersurface in $\mathbf{P}^5$, where we have the notion of Qregularity introduced in [2]. Specifically, $\mathcal{F}$ is Qregular if $H^1(\mathcal{F}(-1)) = H^2(\mathcal{F}(-2)) = H^3(\mathcal{F}(-3)) = 0$ and $H^4(\mathcal{F} \otimes Q(-4)) = H^4(\mathcal{F} \otimes S(-4)) = 0$. In particular, $T_{|G}(-1)$ is G-regular but not Qregular (see Example 2.2), showing that Qregularity is a stronger condition (in fact, it can be proved that Qregularity implies G-regularity).

With our notion of regularity we can prove an analogue of Evans-Griffith theorem, improving the known results (see [10] and [13]) for the total splitting of vector bundles:

Theorem 2.6. *A vector bundle E of rank r on $G = G(1, n)$ splits into a direct sum of line bundles if and only if the following conditions hold:*

(i) $H^1_*(E \otimes Q) = H^2_*(E \otimes S^2Q) = \cdots = H^{n-2}_*(E \otimes S^{n-2}Q) = 0$*;*

(ii) $H^{n-1}_*(E \otimes S^{n-2}Q) = H^n_*(E \otimes S^{n-3}Q) = \cdots = H^{2n-3-i}_*(E \otimes S^iQ) = 0$ *with* $i = \left[\frac{2n-2}{r+1}\right]$.

Proof. It is clear (see Remark 1.1) that a direct sum of line bundles satisfies (i) and (ii), so that we only need to prove the converse. The statement is independent of twists by a line bundle, so that we can assume that E is G-regular but $E(-1)$ is not. In particular, E is globally generated, hence $E \otimes S^iQ(k+1)$ is (very) ample for any $k \geq 0$. This implies, by Le Potier's vanishing theorem,

$$H^j(E \otimes S^iQ(k+1) \otimes \mathcal{O}_G(-n-1)) = 0$$

for $j \geq \operatorname{rank}(E \otimes S^iQ(k+1))$. Hence $H^{2n-3-i}(E \otimes S^iQ(k-n)) = 0$ for $i \leq \frac{2n-2}{r+1} - 1$. This, together with (i) and (ii), implies that $E(-1)$ satisfies all the conditions of G-regularity except the vanishing of $H^{2n-2}(E(-n-1))$, which is therefore different from zero. By Serre's duality, we get $H^0(E^\vee) \neq 0$, which together with the fact that E is generated by its global sections implies that E splits as $E \cong \mathcal{O}_G \oplus E'$. The proof is completed by applying the same technique to E', and making a recursion on the rank. $\square$

Example 2.7. In the particular case $n = 3$ our splitting criterion reads as follows:
Let E be a vector bundle of rank r on the four-dimensional smooth quadric Q_4 such that

$$H^1_*(E \otimes Q) = H^2_*(E \otimes Q) = 0$$

and, only if $r \geq 4$,

$$H^3_*(E) = 0.$$

Then E splits as a direct sum of line bundles.

3 Characterization of the universal bundles on $G(1, n)$

After Theorem 2.6, one could have the temptation of proceeding as in [1], i.e. removing from the statement of the Theorem the conditions not satisfied by the universal bundles and try see whether these fewer conditions characterize direct sums of line bundles and twists of universal bundles. However, this will not work, since Theorem 2.6 already contains few hypotheses. For example, by Remark 1.1, the condition not satisfied by Q is $H^{n-1}_*(Q \otimes S^{n-2}Q) = 0$. However, if we remove that condition, also any S^jQ satisfies the rest of the conditions, so that we cannot hope to characterize the direct sum of line bundles and twist of Q as those bundles E satisfying all the hypotheses of Theorem 2.6 except $H^{n-1}_*(E \otimes S^{n-2}Q) = 0$. This means that we will need to add extra conditions to characterize such direct sums.

We will thus first characterize (with just a finite number of cohomological vanishings) each of the bundles S^jQ or $\bigwedge^j S$. In a final result, we will put all these results together to eventually classify direct sums of line bundles, twists of Q and twists of some $\bigwedge^j S$.

Theorem 3.1. *Let $n \geq 3$ and fix $j \in \{1, \dots, n-2\}$. Let E be a vector bundle on $G = G(1, n)$ such that:*

(i) $H^{n-1-j}(E \otimes S^{n-1-j}Q(-n+j)) \neq 0$;

(ii) $H^1(E(-1)) = H^2(E \otimes Q(-2)) = \cdots = H^{n-1-j}(E \otimes S^{n-2-j}Q(-n+1+j)) = 0$;

(iii) $H^{n-1-j}(E \otimes S^{n-2-j}Q(-n+j)) = H^{n-j}(E \otimes S^{n-3-j}Q(-n+j)) = \cdots = H^{2n-3-2j}(E(-n+j)) = 0$;

(iv) $H^{2n-2-2j}(E(-n-1+j)) = H^{2n-1-2j}(E \otimes Q(-n-2+j)) = \cdots = H^{2n-3-j}(E \otimes S^{j-1}Q(-n)) = 0$;

(v) $H^{2n-2-j}(E \otimes S^{j-1}Q(-n-1)) = H^{2n-1-j}(E \otimes S^{j-2}Q(-n-1)) = \cdots = H^{2n-3}(E(-n-1)) = 0$.

Then E contains $\bigwedge^j S$ as a direct summand. In particular, a vector bundle E of rank $\binom{n-1}{j}$ on G is isomorphic to $\bigwedge^j S$ if and only if it satisfies (i), (ii), (iii), (iv), (v).

Proof. By (i), we can take a nonzero element $\alpha \in H^{n-1-j}(E \otimes S^{n-1-j}Q(-n+j))$. By Serre duality, there exists $\beta \in H^{n-1+j}(E^\vee \otimes S^{n-1-j}Q^\vee(-j-1)) = H^{n-1+j}(E^\vee \otimes S^{n-1-j}Q(-n))$, such that the image of $\alpha \otimes \beta$ in $H^{2n-2}(\mathcal{O}_G(-n-1)) \cong H^{2n-2}(S^{n-1-j}Q \otimes S^{n-1-j}Q^\vee(-n-1))$ is the natural generator (i.e. the dual to the unit of $H^0(\mathcal{O}_G)$). Taking cohomology in $(R^\vee_{n-1-j})$ and tensorizing with $E(-1) \otimes H^{n-1+j}(E^\vee \otimes S^{n-1-j}Q^\vee(-j-1))$ and $S^{n-1-j}Q^\vee(-j)$ we get a commutative diagram:

$$\begin{array}{ccc} H^0(E \otimes \bigwedge^j S^\vee) \otimes H^{n-1+j}(E^\vee \otimes S^{n-1-j}Q(-n)) & \longrightarrow & H^{n-1+j}(\bigwedge^j S^\vee \otimes S^{n-1-j}Q^\vee(-j-1)) \\ \downarrow \sigma \otimes id & & \downarrow \\ H^{n-1-j}(E \otimes S^{n-1-j}Q(-n+j)) \otimes H^{n-1+j}(E^\vee \otimes S^{n-1-j}Q(-n)) & \longrightarrow & H^{2n-2}(\mathcal{O}_G(-n-1)) \end{array}$$

with natural horizontal arrows. We derive from Remark 1.1 that the right arrow is an isomorphism of one-dimensional vector spaces, while condition (ii) implies that σ is an epimorphism. We can thus find $\alpha' \in H^0(E \otimes \bigwedge^j S^\vee)$ such that $\alpha' \otimes \beta$ maps to the unit element in $H^{n-1+j}(\bigwedge^j S^\vee \otimes S^{n-1-j}Q^\vee(-j-1))$.

On the other hand, using Serre duality, the vanishings of (iii) are equivalent, respectively, to

$$H^{n-1+j}(E^\vee \otimes S^{n-2-j}Q(-n+1)) = H^{n-2+j}(E^\vee \otimes S^{n-3-j}Q(-n+2)) = \cdots = H^{2j+1}(E^\vee(-j-1)) = 0.$$

In the same way as above, if we consider sequence $(R^\vee_{n-1-j})$ tensored by $E^\vee(-1)$, this shows that β lifts to an element $\beta' \in H^{2j}(E^\vee \otimes \bigwedge^j S^\vee(-j))$ such that the image of $\alpha' \otimes \beta'$ in $H^{2j}(\bigwedge^j S^\vee \otimes \bigwedge^j S^\vee(-j))$ is the unit element.

Similarly, the vanishings of (iv) are equivalent to

$$H^{2j}(E^\vee(-j)) = H^{2j-1}(E^\vee \otimes Q(-j)) = \cdots = H^{j+1}(E^\vee \otimes S^{j-1}Q(-j)) = 0$$

so, if we consider sequence (R_j) tensored by $E^\vee(-j)$, we see that β' can be lifted to $\beta'' \in H^j(E^\vee \otimes S^jQ(-j))$ such that the image of $\alpha' \otimes \beta''$ in $H^j(\bigwedge^j S^\vee \otimes S^jQ(-j))$ is the unit element.

Finally, the vanishings of (v) are equivalent to

$$H^j(E^\vee \otimes S^{j-1}Q(-j+1)) = H^{j-1}(E^\vee \otimes S^{j-2}Q(-j+2)) = \cdots = H^1(E^\vee) = 0$$

which imply that β'' can be lifted to $\beta''' \in H^0(E^\vee \otimes \bigwedge^j S)$ such that the image of $\alpha' \otimes \beta'''$ in $H^0(\bigwedge^j S^\vee \otimes \bigwedge^j S)$ is the unit element (use sequence $(R^\vee_j)$ tensored by $E^\vee$. But this is nothing but saying that, regarding α' as a morphism $\bigwedge^j S \to E$ and regarding β''' as a morphism $E \to \bigwedge^j S$, their composition is the identity in $\bigwedge^j S$. In other words, $\bigwedge^j S$ is a direct summand of E, as wanted. □

Theorem 3.2. *Let $n \geq 3$ and fix $j \in \{1, \ldots, n-2\}$. Let E be a vector bundle on $G = G(1,n)$ such that:*

(i) $H^{n-1}(E \otimes S^{n-1-j}Q(-n)) \neq 0$

(ii) $H^1(E \otimes S^{j-1}Q(-j)) = \cdots = H^j(E(-j)) = 0;$

(iii) $H^{j+1}(E(-j-1)) = \cdots = H^{n-1}(E \otimes S^{n-2-j}Q(-n+1)) = 0$;

(iv) $H^{n-1}(E \otimes S^{n-2-j}Q(-n)) = \cdots = H^{2n-3-j}(E(-n)) = 0$;

(v) $H^{2n-2-j}(E(-n-1)) = \cdots = H^{2n-3}(E \otimes S^{j-1}Q(-n-j)) = 0$.

Then E contains S^jQ as a direct summand.

Proof. We proceed as in the proof of Theorem 3.1. By condition (i), we can take a nonzero element $\alpha \in H^{n-1}(E \otimes S^{n-1-j}Q(-n))$ and its Serre dual $\beta \in H^{n-1}(E^\vee \otimes S^{n-1-j}Q^\vee(-1)) = H^{n-1}(E^\vee \otimes S^{n-1-j}Q(-n+j))$.

Condition (ii), together with $(R^\vee_{n-1-j})$ tensored by $E(-j-1)$, implies that we can lift α to $\alpha' \in H^{n-1-j}(E \otimes \bigwedge^{n-1-j} S(-j-1)) = H^{n-1-j}(E \otimes \bigwedge^j S^\vee(-j))$. Moreover, (iii) together with (R_j) tensored by $E(-j)$ implies that we can lift α' to $\alpha'' \in H^0(E \otimes S^jQ(-j)) = \mathrm{Hom}(S^jQ, E)$.

On the other hand, writing condition (iv) as

$$H^{n-1}(E^\vee \otimes S^{n-2-j}Q(-n+1-j)) = \cdots = H^{j+1}(E^\vee(-1)) = 0$$

and taking cohomology in $(R^\vee_{n-1-j}$ tensored with $E^\vee(-1)$, we see that we can lift β to $\beta' \in H^j(E^\vee \otimes \bigwedge^{n-1-j} S(-1)) = H^j(E^\vee \otimes \bigwedge^j S^\vee)$. Writing also condition (v) as

$$H^j(E^\vee) = \cdots = H^1(E^\vee \otimes S^{j-1}Q) = 0$$

and taking cohomology in (R_j) tensored with $\mathcal{E}^\vee$ we get that we can lift β' to $\beta'' \in H^0(E^\vee \otimes S^jQ) = \mathrm{Hom}(E, S^jQ)$.

Moreover, α'', β'' are still dual to each other, which means that, regarded as morphisms, there composition is the identity in S^jQ. Hence S^jQ is a direct summand of E. $\square$

Theorem 3.3. *Let E be a vector bundle on $G = G(1,n)$ with $n \geq 3$. Then E is a direct sum of twists of vector bundles of the form $\mathcal{O}_G$, Q or $\bigwedge^j S$ with $j \in \{1, \ldots, n-2\}$ if and only if the following conditions hold:*

(a) $H^1_*(E) = H^2_*(E \otimes Q) = \cdots = H^{n-2}_*(E \otimes S^{n-3}Q) = 0$;

(b) $H^n_*(E \otimes S^{n-3}Q) = \cdots = H^{2n-4}_*(E \otimes Q) = H^{2n-3}_*(E) = 0$;

(c) *for each* $j = 1, \ldots n-2$,
$H^{n-1-j}_*(E \otimes S^{n-2-j}Q) = H^{n-j}_*(E \otimes S^{n-3-j}Q) = \cdots = H^{2n-3-2j}_*(E) =$
$= H^{2n-2-2j}_*(E) = H^{2n-1-2j}_*(E \otimes Q) = \cdots = H^{2n-3-j}_*(E \otimes S^{j-1}Q) = 0$;

(d) $H^2_*(E) = H^3_*(E \otimes Q) = \cdots = H^{n-1}_*(E \otimes S^{n-3}Q) =$
$= H^n(E \otimes S^{n-4}_*Q) = \cdots = H^{2n-4}_*(E) = 0$.

Proof. It follows from Remark 1.1 that a direct sum of twists of $\mathcal{O}_G$, Q or $\bigwedge^j S$ satisfies (a), (b), (c), (d), so that we need to prove the converse. After a twist, we can assume that E is G-regular but $E(-1)$ not. Since $E(-1)$ is not G-regular, and having in mind (b), one of the following conditions is satisfied:

(i) $H^{n-1-j}(E \otimes S^{n-1-j}Q(-n+j)) \neq 0$ for some $j \in \{1, \ldots, n-2\}$;

(ii) $H^{n-1}(E \otimes S^{n-2}Q(-n)) \neq 0$;

(iii) $H^{2n-2}(E(-n-1)) \neq 0$.

In case (i), we are in the hypothesis of Theorem 3.1 (condition (ii) follows from (a), conditions (iii) and (iv) follow from (c) and condition (v) follows from (b)). Hence we can write $E = \bigwedge^j S \oplus E'$ for some other vector bundle E'.

In case (ii), we are in the hypotheses of Theorem 3.2 with $j = 1$ (condition (ii) follows from (a), conditions (iii) and (iv) follow from (d) and condition (v) follows from (b)). We can thus write $E = Q \oplus E'$.

Finally, in case (iii) we have, by Serre duality, $H^0(E^\vee) \neq 0$. Since E is generated by its global sections (by Proposition 2.3), it follows that we can write $E = \mathcal{O}_G \oplus E'$.

In either case, the new vector bundle E' still satisfies the hypotheses (a), (b), (c), (d), so that we can conclude by a recursive argument on the rank. $\square$

Example 3.4. If $n = 3$, the hypotheses of Theorem 3.3 reduce to the fact that E has no intermediate cohomology, and we recover the classification of the ACM bundles on $\mathcal{Q}_4$ proved in [9].

Example 3.5. If $n = 4$, Theorem 3.3 characterizes the direct sums of twists of $\mathcal{O}_G, S, S^\vee$ as those vector bundles E without intermediate cohomology and such that

$$H^2_*(E \otimes Q) = H^3_*(E \otimes Q) = H^4_*(E \otimes Q) = 0$$

so that we recover [1] Theorem 2.4.

Remark 3.6. It is clear that, for example, in order to characterize direct sums of line bundles and twists of Q, we need to remove condition (c) in Theorem 3.3, although we will need more vanishings in (a). Hence in general, we will need a smaller number of conditions to characterize more restrictive bundles.

On the other hand, we could have also proceeded as in Theorem 2.6 and use Le Potier vanishing theorem to improve Theorem 3.3 (or any of the variants we just indicated). We preferred not to do it explicitly, since it represents a small improvement compared with the difficulty to write it in a clear way.

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